NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2631

THE SIMILARITY LAW FOR NONSTEADY HYPERSONIC FLOWS AND REQUIREMENTS FOR THE DYNAMICAL SIMILARITY OF RELATED BODIES IN FREE FLIGHT

By Frank M. Hamaker and Thomas J. Wong

Ames Aeronautical Laboratory
Moffett Field, Calif.

Washington
February 1952

1M

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2631

THE SIMILARITY LAW FOR NONSTEADY HYPERSONIC FLOWS AND REQUIREMENTS FOR THE DYNAMICAL SIMILARITY OF RELATED BODIES IN FREE FLIGHT

By Frank M. Hamaker and Thomas J. Wong

SUMMARY

The similarity law for steady hypersonic flow about slender shapes is extended to nonsteady flows. Similitude for nonsteady flows is found to depend on the same conditions as for steady flows plus additional conditions derived from the nonsteady motions of the bodies under consideration. The aerodynamic forces and moments are correlated for related shapes by means of this law.

Requirements for dynamical similarity of related shapes in free flight, including the correlation of their flight paths, are obtained using the aerodynamic forces and moments as correlated by the hypersonic similarity law. In addition to the conditions of hypersonic similarity, dynamical similarity depends upon conditions derived from the inertial properties of the bodies and the immersing fluids. In order to have dynamical similarity, however, rolling motions in combination with other motions must be eliminated.

INTRODUCTION

The law of similarity for steady hypersonic flows has been studied in some detail. Tsien, in reference 1, derived the law for potential flow about related slender bodies for the two-dimensional and axially symmetrical cases. It was found that similarity of flow exists when the bodies have the same thickness distributions and when the ratio of the free-stream Mach number to the fineness ratio is the same in each flow system. Hayes, in reference 2, showed that the law should remain valid even when the flow includes shock waves and vorticity, and indicated that it should apply to more general three-dimensional flows. In reference 3 the law was derived for flows about slender three-dimensional

shapes in terms of parameters relating the Mach number to the fineness ratio, the aspect ratio, and the angles-of-flight attitude of the bodies. The correlation of force and moment parameters in terms of the similarity parameters was also made for similar flow fields.

The consideration of the problems of free flight suggests the desirability of extending the hypersonic similarity law to cover the case of nonsteady flows. Lin, Reissner, and Tsien, in reference 4, developed necessary conditions for similarity of flow about oscillating two-dimensional bodies in compressible fluids, including flow at hypersonic speeds. An analysis for slender three-dimensional shapes in hypersonic flow is apparently not available, and has therefore been undertaken in the present report following methods similar to those employed in reference 3.

The possibility of obtaining a hypersonic similarity law for correlating the aerodynamic forces and moments on related shapes in free flight suggests a more general dynamical problem, that of correlating their motions with the aid of this law. Hence, it has been undertaken to determine the requirements on the inertial properties of related bodies and the immersing fluids in order that such bodies may exhibit similar free-flight paths, that is, dynamical similarity.

SYMBOLS

a	speed of sound
b	characteristic width or span of body
c	characteristic length or chord of body
C_C	side-force coefficient $\left(\dfrac{\text{side force}}{\frac{1}{2}\rho_o V_o^2 S}\right)$
$\tilde{C}_C$	side-force parameter
C_D	drag coefficient $\left(\dfrac{\text{drag}}{\frac{1}{2}\rho_o V_o^2 S}\right)$
$\tilde{C}_D$	drag parameter

C_l rolling-moment coefficient $\left(\dfrac{\text{rolling moment}}{\frac{1}{2}\rho_o V_o^2 Sb}\right)$

$\tilde{C}_l$ rolling-moment parameter

C_L lift coefficient $\left(\dfrac{\text{lift}}{\frac{1}{2}\rho_o V_o^2 S}\right)$

$\tilde{C}_L$ lift parameter

C_m pitching-moment coefficient $\left(\dfrac{\text{pitching moment}}{\frac{1}{2}\rho_o V_o^2 Sc}\right)$

$\tilde{C}_m$ pitching-moment parameter

C_n yawing-moment coefficient $\left(\dfrac{\text{yawing moment}}{\frac{1}{2}\rho_o V_o^2 Sb}\right)$

$\tilde{C}_n$ yawing-moment parameter

d length of flight path

D displaced-fluid-mass factor $\left(\dfrac{cbt\rho_o}{2}\right)$

f dimensionless perturbation potential function

g dimensionless body shape function

$\bar{i},\bar{j},\bar{k}$ unit vectors along coordinate axes x,y,z, respectively

$\left.\begin{array}{l} I_{x-x} \\ I_{y-y} \\ I_{z-z} \end{array}\right\}$ moments of inertia of body about the x,y,z axes, respectively

$$\left.\begin{array}{l} K_t = M_o\frac{t}{c},\ K_b = M_o\frac{b}{c} \\ K_\alpha = M_o\alpha,\ K_\beta = M_o\beta \\ K_\delta = \delta,\ K_p = M_o\left(\frac{pb}{V_o}\right) \\ K_q = M_o\left(\frac{qc}{V_o}\right),\ K_r = M_o\left(\frac{rc}{V_o}\right) \end{array}\right\}\ \text{hypersonic similarity parameters}$$

$$\left.\begin{array}{l} K_{x-x} = \frac{c^2D}{I_{x-x}},\ K_{y-y} = \frac{c^2D}{I_{y-y}} \\ K_{z-z} = \frac{c^2D}{I_{z-z}},\ K_\mu = \frac{D}{\mu} \end{array}\right\}\ \text{dynamic similarity parameters}$$

l,m,n direction cosines of the unit normal vector to the body surface

M Mach number

M_x,M_y,M_z moments on body about x,y,z axes, respectively

N unit normal vector to surface of body

P fluid static pressure

p,q,r rolling, pitching, and yawing velocities, respectively

P radius of curvature of flight path

$\bar{s}$ vector from the origin of the coordinate system to any point on the body

S characteristic reference area of body ($S = bt$)

t characteristic depth or thickness of body

u,v,w components of body velocity along the x,y,z axes, respectively

V resultant velocity

x,y,z Cartesian coordinates fixed relative to the body

X,Y,Z	forces on body along x,y,z axes, respectively
α	angle of attack
β	angle of sideslip
γ	ratio of the specific heats of the immersing fluid
δ	angle of roll
ξ,η,ζ	dimensionless coordinates corresponding to x,y,z, respectively
θ	time coordinate
μ	mass of body
ρ	density of the fluid
τ	dimensionless time coordinate $\left(\frac{a_o M_o \theta}{c}\right)$
φ	perturbation potential function
Φ	potential function
ω	angular velocity of the body

Subscript

o	free-stream conditions

Superscript

$-$	vector quantities

Except for symbols noted above, all variables used as subscripts indicate partial differentiation with respect to the subscript variable.

DEVELOPMENT AND APPLICATION OF THE SIMILARITY LAW FOR NONSTEADY HYPERSONIC FLOWS

The Basic Law

In general, the analysis for similarity of nonsteady flows parallels that for steady flows (see reference 3), the principle difference being that the nonsteady flow analysis is slightly complicated by the introduction of nonsteady flow terms in the potential and energy equations. Thus, as shown in appendix A, the procedure is to simplify the equations of motion and boundary condition to conform with the restriction to hypersonic flow about slender shapes, and then transform the resulting expressions to dimensionless forms from which the requirements for similarity of flow about related body shapes are determined. Since this analysis is based on the assumption of potential flow, it is desirable to show that the results are not invalidated by the presence of shock waves and vorticity in the flow. To show that this is true, the arguments of Hayes in reference 2 are extended to include nonsteady flows. (See appendix B.)

It is found that similitude depends on the same similarity parameters as for steady motion, namely,

$$K_t = M_o\frac{t}{c} \tag{1}$$

$$K_b = M_o\frac{b}{c} \tag{2}$$

$$K_\alpha = M_o\alpha \tag{3}$$

$$K_\beta = M_o\beta \tag{4}$$

$$K_\delta = \delta \tag{5}$$

plus the following additional parameters arising from the nonsteady motions of bodies:

$$K_p = M_o\left(\frac{pb}{V_o}\right) \tag{6}$$

$$K_q = M_o\left(\frac{qc}{V_o}\right) \tag{7}$$

$$K_r = M_o\left(\frac{rc}{V_o}\right) \tag{8}$$

The statement of the law is equivalent to that for steady flow, namely: The disturbance flow fields about bodies having the same thickness distributions are similar, provided the bodies are undergoing motions such that the same values of the corresponding similarity parameters are obtained.

The new similarity parameters, K_p, K_q, and K_r, can be interpreted in a manner analogous to that for the steady-state parameters. To illustrate, it is recalled that the parameters for hypersonic similarity in steady flow require that the local body slopes with respect to the flow direction at corresponding points on related bodies be inversely proportional to their flight Mach numbers. This statement of requirements applies equally well to nonsteady flows if it is understood that the local slopes include the apparent (or induced) ones as well. In rolling, for example, points on the body surface perform helical motions and the quantity pb/V_o in equation (6) is simply proportional to the slope of the helix with respect to the flow direction. It is thus evident that this slope must also be inversely proportional to the flight Mach number. Similar arguments may be applied to the apparent slopes arising from the pitching and yawing motions, the parenthesized quantities in equations (7) and (8) being proportional to each of these slopes, respectively.

Application of the Law to the Correlation of Aerodynamic Forces and Moments

The correlation of aerodynamic forces and moments on related bodies in unsteady hypersonic flows can be developed by consideration of the static pressure distribution over the bodies. The pressure relation can be obtained from the energy equation (see appendix A, equation (A2)) and is given in the following form:

$$\frac{P}{P_o} = \left[\frac{1 + \frac{\gamma-1}{2a_o^2} V_o^2}{1 + \frac{\gamma-1}{2a^2} (V^2+2\varphi_\theta)}\right]^{\frac{\gamma}{\gamma-1}}$$

When this expression is simplified to conform with the assumptions of the theory and put into dimensionless form, there is obtained the functional relationship (for a constant γ)

$$\frac{P}{P_o} = \frac{P}{P_o}(\xi,\eta,\zeta,\tau;\ K_t,\ K_b,\ K_\alpha,\ K_\beta,\ K_\delta,\ K_p,\ K_q,\ K_r) \qquad (9)$$

As in the case of steady flow, the pressure ratios are the same at corresponding points in similar nonsteady hypersonic flow fields.[1]

The correlation of the aerodynamic forces and moments is then obtained with the aid of relation (9) by integration of the appropriate components of the pressure forces over the related shapes. This correlation can be given in the following forms:

$$\left.\begin{aligned}
M_o C_L &= \tilde{C}_L = \tilde{C}_L(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\
M_o^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\
M_o C_C &= \tilde{C}_C = \tilde{C}_C(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\
M_o C_m &= \tilde{C}_m = \tilde{C}_m(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\
C_n &= \tilde{C}_n = \tilde{C}_n(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r) \\
M_o C_l &= \tilde{C}_l = \tilde{C}_l(K_t, K_b, K_\alpha, K_\beta, K_\delta, K_p, K_q, K_r)
\end{aligned}\right\} \tag{10}$$

The significance of the above equations is that the force and moment parameters are the same for related shapes undergoing motions such that each of the corresponding similarity parameters has the same value.

Caution must be exercised in the use of the correlation equations, however, as there are certain body shapes for which some of these equations are invalid. For example, yawing moment cannot be correlated for thin wings alone because of the moments due to asymmetric drag being of the same order of magnitude as the moments due to cross forces.[2]

DETERMINATION OF THE REQUIREMENTS FOR DYNAMICAL SIMILARITY OF RELATED BODIES AND DISCUSSION OF RESULTS

The requirements for dynamical similarity of related bodies in free flight are now developed with the assumption that the forces and moments on such bodies are correlated by the law of hypersonic similarity. If dynamical similarity is to be coexistent with hypersonic similarity, then the dynamical equations of motion should be transformed to the same dimensionless coordinate system that was used in developing

[1]Analogous statements can be made for the ratios of local temperature, density, and Mach number to the free-stream values.
[2]See reference 3 for details concerning this restriction.

the requirements for hypersonic similarity. In addition, the velocity and force quantities should, of course, be expressed in terms of hypersonic similarity parameters.

In this dynamical system only those forces are considered which correspond to the "power off" condition in free flight. The coordinate axes are taken to be the principal axes of the body so that the products of inertia vanish. The dynamical equations of motion of the body are given by the relations

$$\left.\begin{aligned} u_\theta - rv + qw &= \frac{X}{\mu} \\ v_\theta - pw + ru &= \frac{Y}{\mu} \\ w_\theta - qu + pv &= \frac{Z}{\mu} \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} p_\theta I_{x-x} - qr\left(I_{y-y} - I_{z-z}\right) &= M_x \\ q_\theta I_{y-y} - pr\left(I_{z-z} - I_{x-x}\right) &= M_y \\ r_\theta I_{z-z} - pq\left(I_{x-x} - I_{y-y}\right) &= M_z \end{aligned}\right\} \tag{12}$$

The translational and rotational velocities may be expressed in terms of hypersonic similarity parameters, the Mach number, and the speed of sound of the free stream by the relations

$$\left.\begin{aligned} &u = a_o M_o,\ v = -a_o K_\beta,\ w = a_o K_\alpha \\ &p = a_o \frac{K_p}{b},\ q = a_o \frac{K_q}{c},\ r = a_o \frac{K_r}{c} \end{aligned}\right\} \tag{13}$$

Similarly, the aerodynamic forces and moments are given in terms of the correlation parameters by the relations

$$\left.\begin{aligned}
X &= \frac{\tilde{C}_D}{M_o}\left(\frac{a_o^2\ M_o\ bt\ \rho_o}{2}\right)\\
Y &= \tilde{C}_C\left(\frac{a_o^2\ M_o\ bt\ \rho_o}{2}\right)\\
Z &= \tilde{C}_L\left(\frac{a_o^2\ M_o\ bt\ \rho_o}{2}\right)\\
M_x &= \tilde{C}_l\ b\left(\frac{a_o^2\ M_o\ bt\ \rho_o}{2}\right)\\
M_y &= \tilde{C}_m\ c\left(\frac{a_o^2\ M_o\ bt\ \rho_o}{2}\right)\\
M_z &= \tilde{C}_n\ b\ M_o\left(\frac{a_o^2\ M_o\ bt\ \rho_o}{2}\right)
\end{aligned}\right\} \tag{14}$$

Substituting equations (13) and (14) into equations (11) and (12), and treating only that length of flight path over which M_o can be considered constant, the following set of equations is obtained:

$$K_q\ K_\alpha + K_r\ K_\beta = K_\mu\ \tilde{C}_D \tag{15}$$

$$-\frac{dK_\beta}{d\tau} + K_r - \frac{K_p\ K_\alpha}{K_b} = K_\mu\ \tilde{C}_C \tag{16}$$

$$\frac{dK_\alpha}{d\tau} - K_q - \frac{K_p\ K_\beta}{K_b} = K_\mu\ \tilde{C}_L \tag{17}$$

$$\frac{1}{K_{x-x}\ K_b}\ \frac{dK_p}{d\tau} - \left(\frac{1}{K_{y-y}} - \frac{1}{K_{z-z}}\right)\frac{K_q\ K_r}{M_o^2} = \frac{K_b}{M_o^2}\ \tilde{C}_l \tag{18}$$

$$\frac{1}{K_{y-y}}\ \frac{dK_q}{d\tau} - \left(\frac{1}{K_{z-z}} - \frac{1}{K_{x-x}}\right)\frac{K_r\ K_p}{K_b} = \tilde{C}_m \tag{19}$$

$$\frac{1}{K_{z-z}} \frac{dK_r}{d\tau} - \left(\frac{1}{K_{x-x}} - \frac{1}{K_{y-y}}\right) \frac{K_p K_q}{K_b} = K_b \tilde{C}_n \tag{20}$$

where[3]

$$K_\mu = \frac{D}{\mu} \tag{21}$$

$$K_{x-x} = \frac{c^2 D}{I_{x-x}} \tag{22}$$

$$K_{y-y} = \frac{c^2 D}{I_{y-y}} \tag{23}$$

$$K_{z-z} = \frac{c^2 D}{I_{z-z}} \tag{24}$$

in which

$$D = \frac{cbt\rho_o}{2} \tag{25}$$

The initial conditions to this set of equations are the initial values of the hypersonic similarity parameters.

If both hypersonic similarity and dynamical similarity are to be achieved, it is required that equations (15) through (20) be independent of the Mach number as a separate variable. The elimination of M_o^2 from equation (18) is impossible in the general case, even approximately, because all the terms involved may be of comparable order of magnitude. Consequently, since equation (18) is the relation for rolling effects, it is indicated that flight paths which include rolling cannot be correlated by this method for obtaining dynamical similarity.[4] For motions that do not involve roll, it is seen that dynamical similarity will exist for related shapes if the hypersonic similarity parameters and the dynamical similarity parameters given in equations (21) through (24) have fixed values. These dynamical similarity parameters relate the

---

[3]The parameter K_μ is a familiar stability-analysis term known as the relative mass factor.

[4]For the case of pure rolling, the requirements for similarity of motion can easily be derived using a slightly different set of parameters.

masses of the bodies and the immersing fluids, as well as the distribution of the mass in the body.

A familiar example of motions where rolling effects would be missing is the case of motions confined to the plane of symmetry of the body, the so-called longitudinal motions. To extend the application of this law to the more general case where there are lateral motions as well as longitudinal ones, but no roll, it is necessary to have a suitable symmetry of shape and to have the inertial properties satisfy the relation

$$K_{y-y} = K_{z-z} \tag{26}$$

When these conditions are fulfilled, the flight paths of related bodies can be correlated. As an illustrative example, the disturbed motion of related missile shapes can be examined. The related flight paths will have the same form whether there are stable or unstable oscillations. The lengths of corresponding portions of related flight paths would be proportional to the corresponding lengths of the shapes. This property can be used to relate the amount of damping in the related disturbed flight paths. As shown in appendix C, the radii of curvature at corresponding points of the flight paths would be proportional to the product of the body length and the flight Mach number. Some of these points are illustrated in the example given in figure 1.

CONCLUSIONS

1. The hypersonic similarity law has been found to apply as well for nonsteady flows as for steady flows. The steady-state similarity parameters as well as additional parameters involving the angular velocities must be satisfied.

2. It was found that the motions of related bodies in free flight could be correlated using the hypersonic similarity parameters and additional parameters relating the inertial properties of the bodies and the air densities.

3. The dynamical similarity of the free flight of related bodies can be obtained for motions which include pitching and yawing but no rolling. For pure rolling motions similarity can again be achieved.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Nov. 23, 1951.

APPENDIX A

THE EQUATIONS FOR HYPERSONIC SIMILITUDE

The analysis in this section parallels the analysis for steady flow given in reference 3. Initially, however, angles of roll are not considered. Figure 2 shows the coordinate system and indicates also the free-stream direction and the angular velocities of the body.

The potential equation and the energy equation for nonsteady flow are given in the following relations:

$$\begin{aligned}\Phi_{\theta\theta} &+ \Phi_{xx}(\Phi_x{}^2-a^2) + \Phi_{yy}(\Phi_y{}^2-a^2) + \Phi_{zz}(\Phi_z{}^2-a^2) + \\ &2(\Phi_{xy}\,\Phi_x\,\Phi_y + \Phi_{yz}\,\Phi_y\,\Phi_z + \Phi_{xz}\,\Phi_x\,\Phi_z) + \\ &2(\Phi_x\,\Phi_{x\theta} + \Phi_y\,\Phi_{y\theta} + \Phi_z\,\Phi_{z\theta}) = 0 \qquad \text{(A1)}\end{aligned}$$

$$\Phi_\theta + \frac{1}{2}\left(\Phi_x{}^2 + \Phi_y{}^2 + \Phi_z{}^2\right) + \frac{a^2}{\gamma-1} = \frac{V_o{}^2}{2} + \frac{a_o{}^2}{\gamma-1} \qquad \text{(A2)}$$

Using body axes, the potential function Φ has to the accuracy of this analysis (retaining only terms up to second order) the following perturbation form:

$$\left.\begin{aligned}\Phi_x &= V_o - \frac{V_o\alpha^2}{2} - \frac{V_o\beta^2}{2} + \varphi_x \\ \Phi_y &= -V_o\beta + \varphi_y \\ \Phi_z &= V_o\alpha + \varphi_z \\ \Phi_\theta &= \varphi_\theta\end{aligned}\right\} \qquad \text{(A3)}$$

Eliminating the local sound velocity a between equations (A1) and (A2), substituting equation (A3), and neglecting all terms above the second order, the resulting hypersonic equation is

$$\frac{\varphi_{\theta\theta}}{a_o^2} + M_o^2\varphi_{xx} + \varphi_{yy}\left[M_o^2\beta^2 + (\gamma-1)\frac{M_o}{a_o}\varphi_x - (\gamma+1)\frac{M_o}{a_o}\beta\,\varphi_y + (\gamma-1)\frac{M_o}{a_o}\alpha\,\varphi_z + \right.$$

$$\left.\frac{\gamma+1}{2}\frac{\varphi_y^2}{a_o^2} + \frac{\gamma-1}{2}\frac{\varphi_z^2}{a_o^2} + (\gamma-1)\frac{\varphi_\theta}{a_o^2} - 1\right] +$$

$$\varphi_{zz}\left[M_o^2\alpha^2 + (\gamma-1)\frac{M_o}{a_o}\varphi_x - (\gamma-1)\frac{M_o}{a_o}\beta\,\varphi_y + (\gamma+1)\frac{M_o}{a_o}\alpha\,\varphi_z + \frac{\gamma-1}{2}\frac{\varphi_y^2}{a_o^2} + \right.$$

$$\left.\frac{\gamma+1}{2}\frac{\varphi_z^2}{a_o^2} + (\gamma-1)\frac{\varphi_\theta}{a_o^2} - 1\right] +$$

$$2\left[M_o\,\varphi_{xy}\left(-M_o\beta + \frac{\varphi_y}{a_o}\right) + \varphi_{yz}\left(-M_o^2\alpha\beta - \frac{M_o\beta\varphi_z}{a_o} + \frac{M_o\alpha\varphi_y}{a_o} + \frac{\varphi_y\varphi_z}{a_o^2} + \right.\right.$$

$$\left.M_o\,\varphi_{xz}\left(-M_o\,\alpha + \frac{\varphi_z}{a_o}\right)\right] +$$

$$2\left(\frac{M_o\varphi_{x\theta}}{a_o} - \frac{M_o\beta\varphi_{y\theta}}{a_o} + \frac{\varphi_y\varphi_{y\theta}}{a_o^2} + \frac{M_o\alpha\varphi_{z\theta}}{a_o} + \frac{\varphi_z\varphi_{z\theta}}{a_o^2}\right) = 0 \qquad \text{(A4)}$$

The boundary is given by the body-shape function in nondimensional form

$$g\left(\frac{x}{c}, \frac{y}{b}, \frac{z}{t}\right) = 0 \qquad \text{(A5)}$$

The condition of no normal flow at the surface of the body is given by the vector equation

$$\bar{V} \cdot \bar{N} = 0 \qquad \text{(A6)}$$

where

$$\bar{N} = l\bar{i} + m\bar{j} + n\bar{k} \qquad \text{(A7)}$$

is the unit vector normal to the surface. The condition for slender shapes is given by

$$l \ll 1 \tag{A8}$$

everywhere on the body.

The angular velocities of the body will cause an apparent distortion of the velocity vector at the surface of the body. Expressing the angular velocity in the form

$$\bar{\omega} = p\,\bar{i} + q\,\bar{j} + r\,\bar{k} \tag{A9}$$

the velocity at each point on the surface of the body is given by the vector cross product

$$\bar{\omega} \times \bar{s} = (qz - ry)\,\bar{i} + (rx - pz)\,\bar{j} + (py - qx)\,\bar{k} \tag{A10}$$

The boundary condition (A6) then becomes

$$(\bar{V} - \bar{\omega} \times \bar{s}) \cdot \bar{N} = 0$$

on the surface of the body which, after neglecting higher order terms, becomes

$$V_o g_x - (V_o\beta - \varphi_y + rx - pz)\,g_y + (V_o\alpha + \varphi_z + qx - py)\,g_z = 0 \tag{A11}$$

The other boundary condition is

$$\varphi_x = \varphi_y = \varphi_z = 0 \quad \text{at } x = -\infty \tag{A12}$$

The affine transformation required to obtain the dimensionless form of the equations is given by the following set of relations:

$$\xi = \frac{x}{c},\quad \eta = \frac{y}{b},\quad \zeta = \frac{z}{t},\quad \tau = \frac{\theta a_o M_o}{c} \tag{A13}$$

$$f(\xi, \eta, \zeta, \tau) = \frac{\varphi(x, y, z, \theta)}{a_o M_o c \left(\frac{t}{c}\right)^2} \tag{A14}$$

These are used to transform equations (A4), (A11), and (A12) to the following forms:

$$K_t^2(f_{\tau\tau}+f_{\xi\xi}) + \frac{K_t^2}{K_b^2} f_{\eta\eta}\left[K_\beta^2+(\gamma-1)K_t^2 f_\xi - (\gamma+1)\frac{K_t^2 K_\beta}{K_b} f_\eta + (\gamma-1)K_t K_\alpha f_\zeta + \right.$$

$$\left. \frac{\gamma+1}{2}\frac{K_t^4}{K_b^2} f_\eta^2 + \frac{\gamma-1}{2} K_t^2 f_\zeta^2 + (\gamma-1)K_t^2 f_\tau - 1\right] +$$

$$f_{\zeta\zeta}\left[K_\alpha^2+(\gamma-1)K_t^2 f_\xi - (\gamma-1)\frac{K_t^2 K_\beta}{K_b} f_\eta + (\gamma+1)K_t K_\alpha f_\zeta + \frac{\gamma-1}{2}\frac{K_t^4}{K_b^2} f_\eta^2 + \frac{\gamma+1}{2} K_t^2 f_\zeta^2 + \right.$$

$$\left. (\gamma-1)K_t^2 f_\tau - 1\right] + 2\left[\frac{K_t^2}{K_b} f_{\xi\eta}\left(\frac{K_t^2}{K_b} f_\eta - K_\beta\right) + \frac{K_t}{K_b} f_{\eta\zeta}\left(-K_\alpha K_\beta - K_t K_\beta f_\zeta + \right.\right.$$

$$\left.\left. \frac{K_t^2 K_\alpha}{K_b} f_\eta + \frac{K_t^3}{K_b} f_\eta f_\zeta\right) + K_t f_{\xi\zeta}(K_\alpha + K_t f_\zeta)\right] + 2\left(K_t^2 f_{\xi\tau} - \frac{K_t^2 K_\beta}{K_b} f_{\eta\tau} + \right.$$

$$\left. \frac{K_t^4}{K_b^2} f_\eta f_{\eta\tau} + K_t K_\alpha f_{\zeta\tau} + K_t^2 f_\zeta f_{\zeta\tau}\right) = 0 \tag{A15}$$

$$g_\xi - \left(K_\beta - \frac{K_t^2}{K_b} f_\eta + K_r \xi - K_p \frac{K_t}{K_b}\zeta\right)\frac{g_\eta}{K_b} + (K_\alpha + K_t f_\zeta + K_q \xi - K_p \eta)\frac{g_\zeta}{K_t} = 0 \tag{A16}$$

$$f_\xi = f_\eta = f_\zeta = 0 \quad \text{at } \xi = -\infty \tag{A17}$$

The transformed differential equation of motion, the transformed boundary conditions, and the following parameters form the hypersonic similarity law for nonsteady flows:

$$\left.\begin{aligned} K_t &= M_o \frac{t}{c}, & K_p &= M_o\left(\frac{pb}{V_o}\right) \\ K_b &= M_o \frac{b}{c}, & K_q &= M_o\left(\frac{qc}{V_o}\right) \\ K_\alpha &= M_o \alpha, & K_r &= M_o\left(\frac{rc}{V_o}\right) \\ K_\beta &= M_o \beta \end{aligned}\right\} \tag{A18}$$

Angles of roll were not included in this derivation as they unnecessarily complicate the algebra. Had they been included, however, the result would be the same as above with the additional requirement, as found in reference 3, that for flows over related bodies to be similar the angles of roll must be the same. Hence the additional hypersonic similarity parameter is

$$K_\delta = \delta \tag{A19}$$

It is of interest to note that the parameters for the angular velocities may be derived by differentiation of the steady-state similarity parameters by the dimensionless time variable, thus:

$$\left.\begin{aligned}\frac{\partial K_\alpha}{\partial \tau} &= K_q \\ \frac{\partial K_\beta}{\partial \tau} &= K_r \\ \frac{\partial K_\delta}{\partial \tau} &= \frac{K_p}{K_b}\end{aligned}\right\} \tag{A20}$$

APPENDIX B

EXTENSION OF POTENTIAL FLOW ANALYSIS TO NONISENTROPIC FLOW

In reference 3 the similarity law was extended to nonisentropic steady flow by using the conclusions of Hayes in reference 4. The essential point of his analysis is as follows: If the transformation

$$x = V_o \psi = M_o a_o \psi \tag{B1}$$

is used on the equation for steady-state hypersonic flow in perturbation form[5]

$$M_o \varphi_{xx} - \left[1 - (\gamma-1)\frac{M_o}{a_o}\varphi_x - \frac{\gamma+1}{2}\frac{{\varphi_y}^2}{{a_o}^2} - \frac{\gamma-1}{2}\frac{{\varphi_z}^2}{{a_o}^2}\right]\varphi_{yy} -$$

$$\left[1 - (\gamma-1)\frac{M_o}{a_o}\varphi_x - \frac{\gamma-1}{2}\frac{{\varphi_y}^2}{{a_o}^2} - \frac{\gamma+1}{2}\frac{{\varphi_z}^2}{{a_o}^2}\right]\varphi_{zz} +$$

$$2\left(\frac{M_o}{a_o}\varphi_y\varphi_{xy} + \frac{\varphi_y\varphi_z\varphi_{yz}}{{a_o}^2} + \frac{M_o}{a_o}\varphi_z\varphi_{xz}\right) = 0 \tag{B2}$$

there is obtained the equation

$$\frac{\varphi_{\psi\psi}}{{a_o}^2} - \left[1 - (\gamma-1)\frac{\varphi_\psi}{{a_o}^2} - \frac{\gamma+1}{2}\frac{{\varphi_y}^2}{{a_o}^2} - \frac{\gamma-1}{2}\frac{{\varphi_z}^2}{{a_o}^2}\right]\varphi_{yy} -$$

$$\left[1 - (\gamma-1)\frac{\varphi_\psi}{{a_o}^2} - \frac{\gamma-1}{2}\frac{{\varphi_y}^2}{{a_o}^2} - \frac{\gamma+1}{2}\frac{{\varphi_z}^2}{{a_o}^2}\right]\varphi_{zz} +$$

$$2\left(\frac{\varphi_y\varphi_{\psi y}}{{a_o}^2} + \frac{\varphi_y\varphi_z\varphi_{yz}}{{a_o}^2} + \frac{\varphi_z\varphi_{\psi z}}{{a_o}^2}\right) = 0 \tag{B3}$$

[5]In all the equations of this section, the wind axes are made to coincide with the body axes in order not to obscure the argument.

This is the exact two-dimensional potential equation for nonsteady flow provided ψ is the time coordinate. In this transformed equation φ becomes a complete potential function instead of a perturbation one and a_o is the sound velocity for no flow. The boundary, which was a surface in the x,y,z space, is now a time dependent curve in the y,z plane corresponding to the outline of the cross sections of the body. The point of view of the observer in this representation of hypersonic flow is that of moving with the fluid over the boundary or surface of the body. The result of the hypersonic approximation is that the only disturbances seen are those propagated outwards from the body. Disturbances upstream and downstream of the observer are ignored because of the small magnitude of the velocity of sound compared to the stream velocity. Hayes shows that shock waves and nonisentropic conditions do not affect this point of view, so long at the local Mach number is large compared to 1.

To apply these concepts to this report, the nonsteady part of the flow over the slender body may be considered, in the transformed representation, as simply an additional nonsteady increment on the already nonsteady boundary. In fact, this can be demonstrated analytically by applying equation (B1) to the nonsteady flow equation:

$$\varphi_{\theta\theta} + M_o^2\varphi_{xx} - \left[1 - (\gamma-1)\frac{M_o}{a_o}\varphi_x - \frac{\gamma+1}{2}\frac{\varphi_y^2}{a_o^2} - \frac{\gamma-1}{2}\frac{\varphi_z^2}{a_o^2} - (\gamma-1)\frac{\varphi_\theta}{a_o^2}\right]\varphi_{yy} -$$
$$\left[1 - (\gamma-1)\frac{M_o}{a_o}\varphi_x - \frac{\gamma-1}{2}\frac{\varphi_y^2}{a_o^2} - \frac{\gamma+1}{2}\frac{\varphi_z^2}{a_o^2} - (\gamma-1)\frac{\varphi_\theta}{a_o^2}\right]\varphi_{zz} +$$
$$2\left(\frac{M_o}{a_o}\varphi_y\ \varphi_{xy} + \frac{M_o}{a_o}\varphi_z\ \varphi_{xz} + \frac{\varphi_y\ \varphi_z}{a_o^2}\varphi_{yz}\right) +$$
$$2\left(\frac{M_o}{a_o}\varphi_{x\theta} + \frac{\varphi_y}{a_o^2}\varphi_{y\theta} + \frac{\varphi_z}{a_o^2}\varphi_{z\theta}\right) = 0 \qquad \text{(B4)}$$

with an additional variable change of

$$\Omega = \theta + \psi \qquad \text{(B5)}$$

obtaining, thereby, the same equation (B3) with ψ replaced by Ω. Hence Hayes' conclusions concerning steady-state flow should apply equally well to nonsteady flows.

APPENDIX C

CORRELATION OF THE FLIGHT PATH CURVATURE

Consider related bodies moving through properly related fluids in paths of finite radius of curvature. Equating the centrifugal force to the side force, the following relation is obtained:

$$\mu \frac{V_o^2}{R} = C_C \frac{1}{2} \rho_o V_o^2 S \qquad \text{(C1)}$$

After rearranging in terms of similarity parameters, equation (C1) becomes

$$\frac{M_o c}{R} = \widetilde{C}_C K_\mu \frac{S}{bt} = \text{constant} \qquad \text{(C2)}$$

The parameter $M_o c/R$ correlates the radii of curvature at corresponding points of similar flight paths.

REFERENCES

1. Tsien, Hsue-Shen: Similarity Laws of Hypersonic Flows. Jour. Math. and Phys., vol. 25, no. 3, Oct. 1946, pp. 252-259.

2. Hayes, Wallace D.: On Hypersonic Similitude. Quart. Appl. Math. vol. V, no. 1, April 1947, pp. 105-106.

3. Hamaker, Frank M., Neice, Stanford E., and Eggers, A. J., Jr.: The Similarity Law for Hypersonic Flows About Slender Three-Dimensional Shapes. NACA TN 2443, 1951.

4. Lin, C. C., Reissner, E., and Tsien, H. S.: On Two-Dimensional Non-Steady Motion of a Slender Body in a Compressible Fluid. Jour. Math. and Phys., vol. 27, no. 3, Oct. 1948, pp. 220-231.

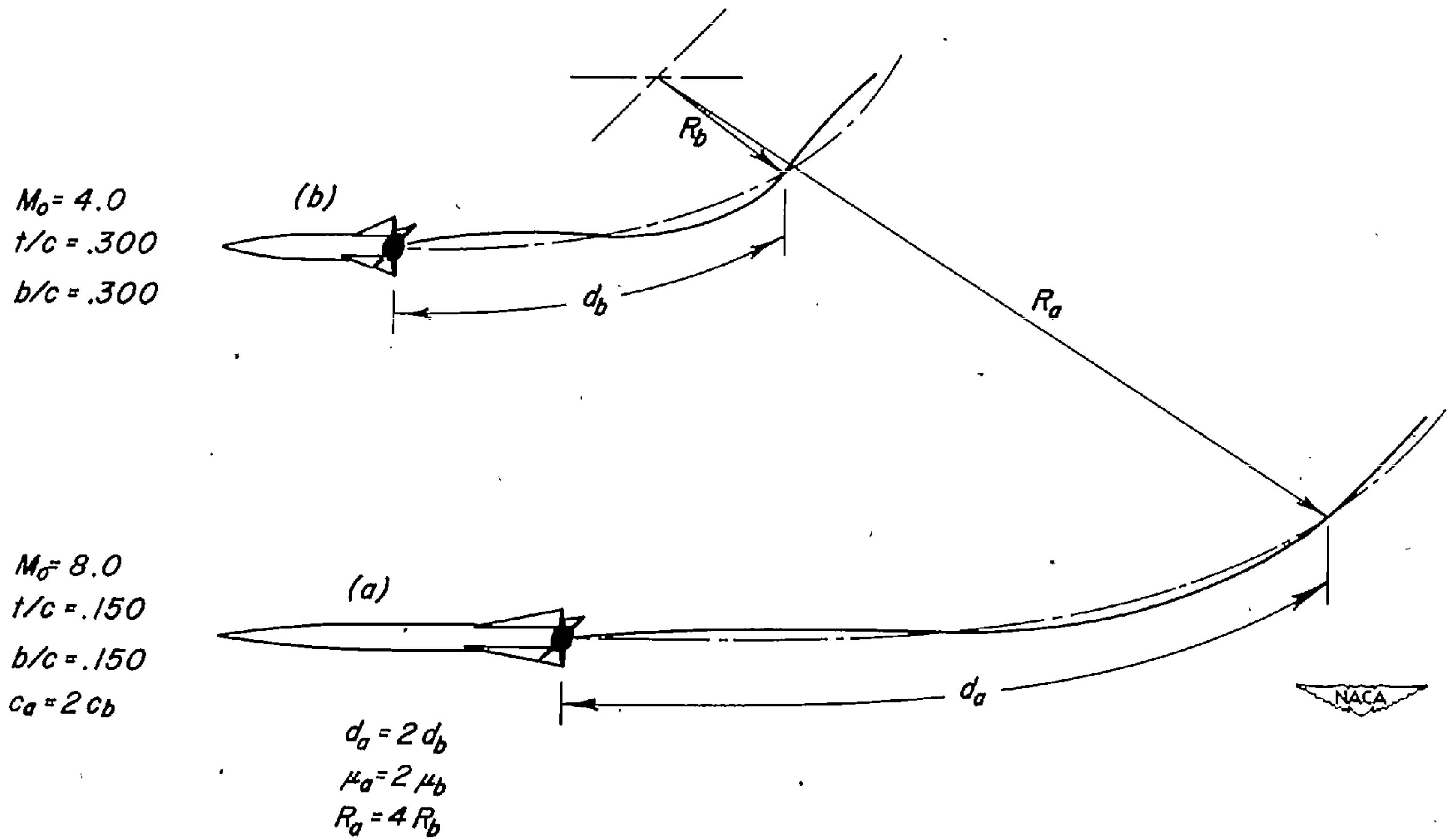

Figure 1.- Related wing-body combinations at hypersonic speeds.

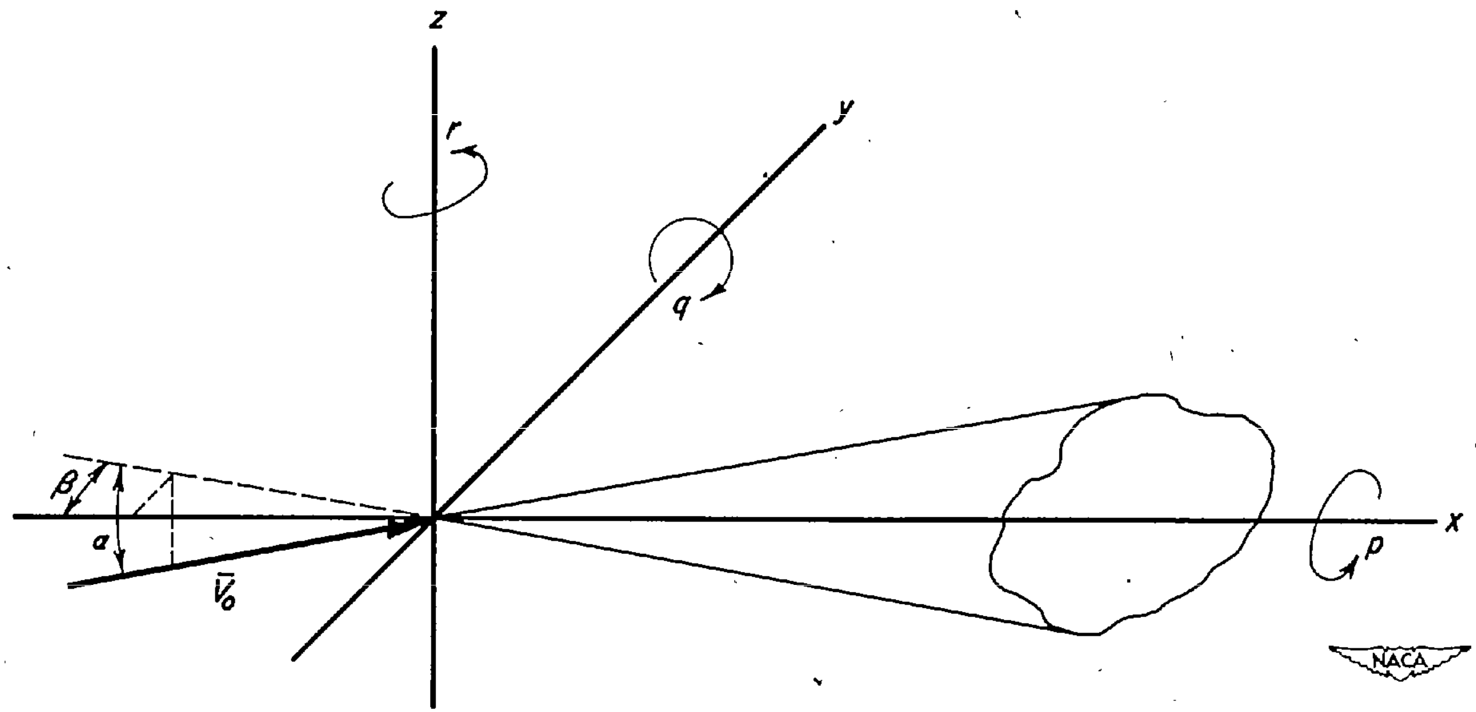

Figure 2.– Schematic diagram of orientation of body in flow.

NACA-Langley - 2-12-52 - 1000